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Ex Roll a dice
Let X be the result.

ECS315 2014/1 Part III.1 Dr.Prapun

7 Random variables

deviates

In performing a chance experiment, one is often not interested in the particular outcome that occurs but in a specific numerical value associated with that outcome. In fact, for most applications, measurements and observations are expressed as numerical quantities.

Example 7.1. Take this course and observe your grades.

$\Omega = \{A, B+, B, C+, C, D+, D, F\}$

RV

Define a function $G(\cdot)$ that transforms the outcome into a real number:

$G(A) = 4$ $G(B+) = 3.5$ $G(C+) = 2.5$ $G(D+) = 1.5$ $G(F) = 0$

$G(B) = 3$ $G(C) = 2$ $G(D) = 1$

7.2. The advantage of working with numerical quantities is that we can perform mathematical operations on them.

add, subtract, multiply, divide, average, max, min, etc.

In the mathematics of probability, averages are called expectations or expected values.

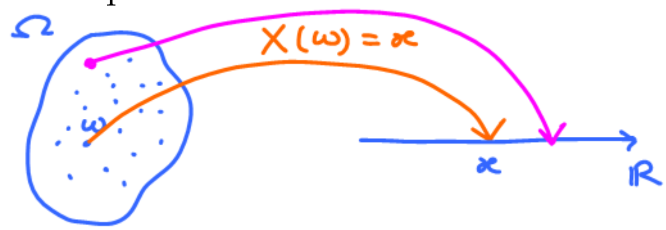
① Intuitive definitions

② Formal definitions

Definition 7.3. A 2.1 real-valued function $X(\omega)$ defined for all points ω in a sample space Ω is called a ***random variable*** (r.v. or RV) [27].

- So, a random variable is a 2.2 rule that assigns a numerical value to each possible outcome of a chance experiment.

$X : \Omega \to \mathbb{R}$



- Intuitively, 1.1 a random variable is a variable that takes on its values by chance. 1.2 Random variables are (numerical) quantities whose values are determined by the results of random experiment.
- The convention is to use capital letters such as X, Y, Z to denote random variables.

Example 7.4. Roll a fair dice: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

These functions are random variables.

$X(\omega) = \omega$

$Y(\omega) = (\omega - 3)^2$

$Z(\omega) = \sqrt{Y(\omega)}$

$$U(\omega) = \begin{cases} 1, & \omega \geq 3 \\ 0, & \omega < 3 \end{cases}$$

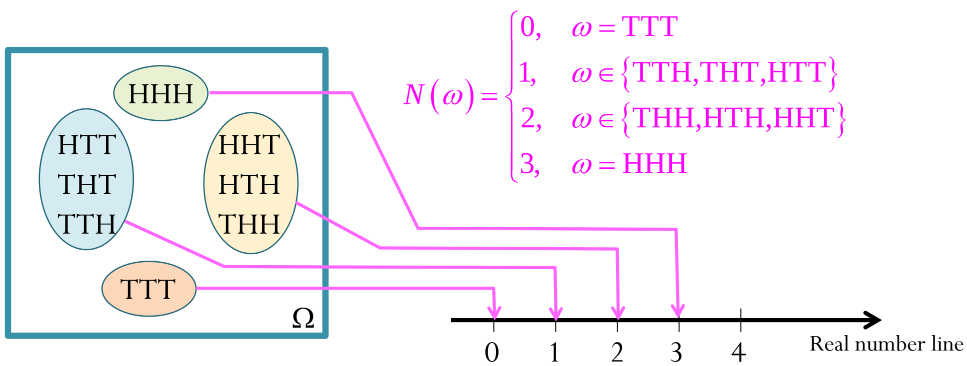
Note that more than one RVs can be defined on one sample space.

[27] The term "random variable" is a misnomer. Technically, if you look at the definition carefully, a random variable is a deterministic function; that is, it is not random and it is not a variable. [Toby Berger][25, p 254]

- As a function, it is simply a rule that maps points/outcomes ω in Ω to real numbers.
- It is also a deterministic function; nothing is random about the mapping/assignment. The randomness in the observed values is due to the underlying randomness of the argument of the function X, namely the experiment outcomes ω.
- In other words, the randomness in the observed value of X is induced by the underlying random experiment, and hence we should be able to compute the probabilities of the observed values in terms of the probabilities of the underlying outcomes.

Example 7.5 (Three Coin Tosses)**.** Counting the number of heads in a sequence of three coin tosses.

$$\Omega = \{\text{TTT,TTH,THT,THH,HTT,HTH,HHT,HHH}\}$$



Example 7.6 (Sum of Two Dice)**.** If S is the sum of the dots when rolling one fair dice twice, the random variable S assigns the numerical value $i+j$ to the outcome (i, j) of the chance experiment.

Example 7.7. Continue from Example 7.4,

(a) What is the probability that $X = 4$?

$X(\omega) = 4$ occurs when $\omega = 4$

Hence, $P[X = 4] = P(\{4\}) = \frac{1}{6}$

Note the use of square brackets

(b) What is the probability that $Y = 4$?

$Y(\omega) = 4$ occurs when $(\omega - 3)^2 = 4 = 2^2$

$\omega - 3 = \pm 2$

$\omega = 3 \pm 2 = 1$ or 5

Hence, $P[Y = 4] = P(\{1, 5\}) = P(\{1\}) + P(\{5\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

Definition 7.8. Events involving random variables:

- [some condition(s) on X] = the set of outcomes in Ω such that $X(\omega)$ satisfies the conditions.
- $[X \in B] = \{\omega \in \Omega : X(\omega) \in B\}$
- $[a \le X < b] = [X \in [a, b)] = \{\omega \in \Omega : a \le X(\omega) < b\}$
- $[X > a] = \{\omega \in \Omega : X(\omega) > a\}$
- $[X = x] = \{\omega \in \Omega : X(\omega) = x\}$
    - We usually use the corresponding lowercase letter[28] to denote
        (a) a possible value (realization) of the random variable
        (b) the value that the random variable takes on
        (c) the running values for the random variable

$[X = c]$
$[X = 5]$
$[X = \pi]$
$[X = 2]$
$\vdots$

All of the above items are sets of outcomes. They are all events!

Example 7.9. Continue from Examples 7.4 and 7.7,

(a) $[X = 4] = \{\omega : X(\omega) = 4\}$

(b) $[Y = 4] = \{\omega : Y(\omega) = 4\} = \left\{\omega : (\omega - 3)^2 = 4\right\}$

7.10. Event of the form "[some condition(s) on X]" or "[some statement(s) about X]" can be written in the from $[X \in B]$ for some appropriate B.

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[28] This is the same as writing $[X = c]$ where c is a constant. Basically, it is a generic notation for $[X = 5]$, $[X = 1.6]$, $[X = \pi]$, etc. We use this when

(a) we don't want to specify the constant in the expression yet or

(b) we want to say that the statement/equation/property containing it is valid for any value of c.

It turns out that, later on, we will have to deal with many random variables and hence it is convenient to have the name of the constant c match the name of the corresponding random variable. So, we talk about the events $[X = x]$, $[Y = y]$, and $[Z = z]$ instead of having to find new name for the constant corresponding to each one of them, say, $[X = c]$, $[Y = d]$, and $[Z = h]$.

You may think we can use constants $c_1, c_2, \ldots$. However, we also will have to deal with ransom variables $X_1, X_2, \ldots, Y_1, Y_2, \ldots, Z_1, Z_2, \ldots$. So, again, will have to come up with new names for a lot of constants.

Example 7.11. Express each event below in the form $[X \in B]$.

(a) $[5 \le X < 8] = [X \in [5,8)]$ $\qquad$ $B = [5,8)$

(b) $[|X| < 3] = [-3 < X < 3] = [X \in (-3,3)]$ $\qquad$ $B = (-3,3)$

(c) $[X > 2] = [X \in (2,\infty)]$ $\qquad$ $B = (2,\infty)$

(d) $[X = 1] = [X \in \{1\}]$ $\qquad$ $B = \{1\}$

Definition 7.12. We also have another notation for $P[X \in B]$:

$$P^X(B) \equiv P[X \in B].$$

Observe that this function P^X is a set function. It maps subsets of real numbers into their probability values. Technically, we call this function the **law** or **distribution** of the random variable X. However, later on, we shall see that there are many functions that are also referred to as the "distribution" of X as well. They are all equivalent in the sense that they (almost surely) give the same information about probability concerning X.

Definition 7.13. To avoid double use of brackets (round brackets over square brackets), we write $P[X \in B]$ when we means $P([X \in B])$. Hence,

$$P[X \in B] = P(\overbrace{[X \in B]}^{\text{An event as defined in Defn. 7.8}}) = P(\{\omega \in \Omega : X(\omega) \in B\}).$$

Probability measure (as defined in Section 5)

Similarly,

$$P[X < x] = P([X < x]) = P(\{\omega \in \Omega : X(\omega) < x\}).$$

Example 7.14. In Example 7.5 (Three Coin Tosses), if the coin is fair, then

$$P[N < 2] \equiv P([N < 2]) = P(\{\omega : N(\omega) < 2\})$$
$$= P(\{TTT, TTH, THT, HTT\}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

$$P[N = 3] \equiv P([N = 3]) = P(\{\omega : N(\omega) = 3\}) = P(\{HHH\}) = \frac{1}{8}$$

(Now)

7.15. At a certain point in most probability courses, the sample space is rarely mentioned anymore and we work directly with random variables. The sample space often "disappears" along with the "(ω)" of $X(\omega)$ but they are really there in the background.

Definition 7.16. A set S is called a ***support*** of a random variable X if $P[X \in S] = 1$.

- To emphasize that S is a support of a particular variable X, we denote a support of X by S_X.
- Practically, we define a support of a random variable X to be the set of all the "possible" values of X.[29]
- For any random variable, the set $\mathbb{R}$ of all real numbers is always a support; however, it is not that useful because it does not further limit the possible values of the random variable.
- Recall that a support of a probability measure P is any set $A \subset \Omega$ such that $P(A) = 1$.

Definition 7.17. The ***probability distribution*** is a description of the probabilities associated with the random variable.

7.18. There are three types of of random variables. The first type, which will be discussed in Section 8, is called ***discrete random variable***. To tell whether a random variable is discrete, one simple way is to consider the "possible" values of the random variable. If it is limited to only a finite or countably infinite number of possibilities, then it is discrete. We will later discuss ***continuous random variables*** whose possible values can be anywhere in some intervals of real numbers.

[29]Later on, you will see that 1) a default support of a discrete random variable is the set of values where the pmf is strictly positive and 2) a default support of a continuous random variable is the set of values where the pdf is strictly positive.